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ASYMPTOTIC REDUCTION OF SOLUTION SPACE DIMENSION FOR DYNAMICAL SYSTEMS

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Abstract. We introduce the equivalence relation in the solution space to initial value problem for dynamical systems: the distance between their trajectories approaches zero with time approaching infinity. The phenomenon "the dimension of the quotient space is less than one of the initial spaces" is named "asymptotic reduction of solution space dimension". We demonstrate that various well-known results including existence of special solutions of delay differential equations with small argument can be presented uniformly by this method. These results are extended to operator-difference equations and improved by the new method of splitting spaces. Some results are further verified by computations.

Keywords: difference equation, delay differential equation, asymptotic, quotient space, special solution.

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1. Introduction

One of the main problems in the theory of dynamical systems is the problem of solution behavior of IVP as time approaches infinity. Many mathematical methods were developed to solve this problem including the theory of stability, the method of characteristic equations for autonomous and periodic dynamic systems, method of special solutions for delay differential equations. Various sufficient conditions were obtained to provide some types of solution behavior. Various definitions and notations were introduced for each.

We introduce the new definitions to unify ones proposed earlier, to present previous and to obtain new results.

Remark 1.1. With wide spread of computers the following auxiliary methods appeared: exact (if possible) or approximate solving of standard problems by mathematical software packages we used them [8]; conducting of experiments to hypothesize (we used it [11]); validating computations if arising conditions are too complicated to obtain explicit estimations by hand (see below Section 6).

Section 2 contains definitions of asymptotic equivalence, λ-exponential asymptotic equivalence and the phenomenon of asymptotic dimension reduction in various types of solution spaces to dynamical systems.

Section 3 demonstrates that the introduced definitions unify some known results on asymptotic behavior of solutions to dynamical systems and are more general than ones proposed earlier.

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Section 4 classifies and describes special properties of solutions of delay differential equations: existence of special (slowly varying) solutions; asymptotical approximation of all solutions by special solutions.

Section 5 introduces classes of IVP for operator difference equations using the method of splitting spaces and defines conditions for special properties of their solutions.

Section 6 contains new results on asymptotic solution behavior of IVP for delay differential equations using the results of Section 5.

2. Main definitions

We consider the dynamical systems as equations for functions depending on time satisfying the property "the present depends on the past only" (differential equations, delay differential equations, Volterra integral equations of the second kind, difference equations etc.). "Ordinary" equations in more general form can be presented as follows (we are restricted to existence and uniqueness of solution of IVP).

Remark 2.1.. There are many definitions of dynamical systems given not as "equations" but as "sets of solutions" or as "flows" in the references. Our approach is close, for instance, to [1].

Definition 2.1. A dynamical system is a tuple consisting of

a number $h \geq 0$ [index of delay];

a totally ordered set Λ of real numbers with the least element but without the greatest one [domain of functions]: $\Lambda = \mathbb{R}_h := [-h, \infty)$ or $\Lambda = \mathbb{N}_0 := \{0, 1, 2, ...\}$;

a topological space Z [range of functions];

a set Φ of functions $[-h, 0] \to Z$ [initial conditions]; if $h = 0$ then $\Phi = Z$;

a function $W(t, \varphi) : \Lambda \times \Phi \to Z$ such that its restriction on $[-h, 0]$ equals φ [solutions of initial value problems].

If $\Lambda = \mathbb{R}_h$ then $W(t, \varphi)$ is supposed to be continuous with respect to t.

As usually, if Z is a linear space and $W(t, \alpha_1\varphi_1 + \alpha_2\varphi_2) \equiv \alpha_1 W(t, \varphi_1) + \alpha_2 W(t, \varphi_2)$ then the dynamical system is said to be linear.

If $h > 0$ then we have equations with bounded delay; if $\Lambda = \mathbb{N}_0$ then the term "difference equations" is used.

We will consider the following classes of spaces with their dimensions:

1-spaces: $Z = \mathbb{R}$; dimension = 1;

d-spaces: $\Phi = Z = \mathbb{R}^d, d \in \mathbb{N} := \{1, 2, ...\}$; dimension = d;

N-spaces: Z is a normed linear space with norm $|| \cdot ||_Z$; dimension (finite or infinite) is the number of elements in the basis;

M-spaces: Z is a metric space with metric $\rho_Z(\cdot, \cdot)$; the inductive Ind-dimension is used;

U-spaces: Z is a uniform space with set of entourages Υ_Z; Ind-dimension is used;

T-spaces: Z is a topological space; Ind-dimension is used.

If $h = 0$ then dimension of Φ equals one of Z.

We give the well-known definitions in our notations for comparison:

Definition 2.2. [3] (for *d-Spaces*):

2.1. If for any $\varphi_0 \in \Phi$

$$(\forall \varepsilon > 0)(\exists \delta > 0)((||\varphi - \varphi_0|| < \delta) \Rightarrow (\forall t \in \Lambda)(||W(t, \varphi) - W(t, \varphi_0)|| < \varepsilon)) \qquad (1)$$

then the solution $W(t, \varphi_0)$ is said to be stable.

2.2. If additionally $\lim\{||W(t, \varphi) - W(t, \varphi_0)|| : t \to \infty\} = 0$

then the solution $W(t,\varphi_0)$ is said to be asymptotically stable.

2.3. If additionally $(\exists\lambda>0)(\forall\varphi)(\exists c>0)(||W(t,\varphi)-W(t,\varphi_0)||\le c\exp(-\lambda t))$

then the solution $W(t,\varphi_0)$ is said to be asymptotically exponentially stable [we will call it "λ-stable"].

2.4. If there exists such d^*-dimensional domain $\Phi'\subset\Phi, \Phi'\ne\Phi, d^*\le d$ that

$(\forall\varphi_0\in\Phi')(\forall\varepsilon>0)(\exists\delta>0)(\forall\varphi\in\Phi)\big((||\varphi-\varphi_0||<\delta)\Rightarrow(\forall t\in\Lambda)(||W(t,\varphi)-W(t,\varphi_0)||<\varepsilon)\big)$

then the solution $W(t,\varphi_0)$ is said to be conditionally stable [we will also add: $(d\searrow d^*)$]; if also (1) fulfils then the solution $W(t,\varphi_0)$ is said to be asymptotically conditionally stable.

These definitions are naturally extended to *N-spaces*, *M-spaces* and, except "λ-stable", to *U-spaces*. They can be extended to *T-spaces* only in the case $W(t,\varphi_0)\equiv const=\varphi_0$ because neighbors of different points are incomparable.

We propose more general definitions.

Definition 2.3. The following equivalence is said to be *asymptotic equivalence* (λ-*exponential asymptotic equivalence*, $\lambda>0$ respectively) in the solution space S_w.

2.1. For *N-Spaces*:

$$(\varphi_1\sim\varphi_2)\Leftrightarrow(\lim\{||W(t,\varphi_1)-W(t,\varphi_2)||_Z:t\to\infty\}=0);$$

$$(\varphi_1\sim_\lambda\varphi_2)\Leftrightarrow(\exists\gamma>0)(\forall t\in\Lambda)(||W(t,\varphi_1)-W(t,\varphi_2)||_Z\le\gamma\exp(-\lambda t))$$

respectively.

2.2. For *M-Spaces*:

$$(\varphi_1\sim\varphi_2)\Leftrightarrow(\lim\{\rho_Z(W(t,\varphi_1),W(t,\varphi_2)):t\to\infty\}=0);$$

$$(\varphi_1\sim_\lambda\varphi_2)\Leftrightarrow(\exists\gamma>0)(\forall t\in\Lambda)(\rho_Z(W(t,\varphi_1),W(t,\varphi_2))\le\gamma\exp(-\lambda t))$$

respectively.

2.3. For *U-Spaces*:

$$(\varphi_1\sim\varphi_2)\Leftrightarrow((\forall V\in\Upsilon_Z)(\exists t_1\in\Lambda)(\forall t>t_1)((W(t,\varphi_1),W(t,\varphi_2))\in V)).$$

(λ-exponential asymptotic equivalence cannot be defined in such general spaces).

2.4. For *T-Spaces* an additional construction is necessary. Let F be a filter of sets of Z such that (probably, except one point) $(\forall z\in Z)(\exists V\in F)(z\notin V)$.

The following equivalence is said to be *asymptotic equivalence with respect to the filter* F:

$$(\varphi_1\sim_F\varphi_2)\Leftrightarrow((\forall V\in F)(\exists t_1\in\Lambda)(\forall t>t_1)((W(t,\varphi_1)\in V)\wedge(W(t,\varphi_2))\in V)).$$

Remark 2.2. In some papers the term "asymptotic equivalence" is understood as proximity between solutions of different dynamical systems with the same space Φ. For instance [2], in our notations $(W_1(t,\varphi)\approx W_2(t,\varphi))\Leftrightarrow(\lim\{||W_1(t,\varphi)-W_2(t,\varphi)||_Z:t\to\infty\}=0)$.

Definition 2.4. The quotient space $\Phi^*:=\Phi/\sim$ of the space Φ by the asymptotic equivalence is said to be an *asymptotic quotient space*; respectively, the quotient space $\Phi^*_\lambda=\Phi/\sim_\lambda$ of the space Φ by the λ-exponential asymptotic equivalence is said to be λ-*exponential asymptotic quotient space*.

Obviously, linear structures of spaces Φ are transferred to quotient spaces Φ^* in a natural way. On the contrary, metric and uniform structures are not transferred in general case.

Example 2.1. *d-Spaces*, $d=2:\Phi=Z=\{(\varphi_1,\varphi_2)\}=\mathbb{R}^2$,
$W(t,\varphi_1,\varphi_2)=(\varphi_1\cdot\exp(-t)+(1-exp(-t))\cdot\cos(\varphi_2^2),\varphi_2\cdot\exp(-t))$.
We have $\lim\{W(t,\varphi_1,\varphi_2):t\to\infty\}=(\cos(\varphi_2^2),0)$. Hence, there are two different classes of asymptotic equivalence in Φ^*:

$C_1:=\{(\varphi_1\in\mathbb{R},\varphi_2=\pm\sqrt{2k\pi}):k\in\mathbb{N}_0\}$, $\lim\{W(t,\varphi_1,\varphi_2):t\to\infty\}=(1,\ 0)$;

$C_2 := \{(\varphi_1 \in \mathbb{R}, \varphi_2 = \pm\sqrt{2k\pi+\pi}) : k \in \mathbb{N}_0\},\ \lim\{W(t,\varphi_1,\varphi_2) : t \to \infty\} = (-1,\ 0)$.

The difference $|\sqrt{2k\pi+\pi} - \sqrt{2k\pi}|$ is arbitrary small for arbitrary large k, i. e. the Hausdorff distance between classes C_1 and C_2 equals zero. Hence, there cannot be any natural metric in the quotient space Φ^*.

Definition 2.5. If the dimension d^* of Φ^* (of Φ^*_λ respectively) is less than one d of Φ then it is said to be *the phenomenon of asymptotic reduction of dimension* (PARD or PARD$_\lambda$ respectively $(d \searrow d^*)$) of space of solutions of initial value problems for a dynamical system.

If PARD$_\lambda$ occurs and there exists such $\lambda_1 > \lambda$ that PARD$_{\lambda_1}$ occurs for Φ^*_λ then multiple PARD$_{\lambda,\lambda_1}$ occurs.

3. Review of some known results with respect to new definitions

Lemma 3.1. For *d-Spaces*: If the dynamical system is linear and $h = 0$ then PARD$(d \searrow 0)$ implies asymptotic stability of the zero solution.

Proof. Let $\{e_1, ..., e_d\}$ be a basis in $\mathbb{Z}$. As all norms in $\mathbb{R}^d$ are equivalent, we will use the norm $||\sum_{j=1}^d x_j e_j||_0 := \max\{|x_j| : j = 1, ..., d\}$.

Choose a small number $\varepsilon > 0$. Due to Definition 2.3 and linearity,

$(\forall j \in \{1, ..., d\})(\lim\{||W(t,e_j) - 0||_0 : t \to \infty\} = 0)$.

Denote $M_j := \sup\{||W(t,e_j)||_0 : t \in \Lambda\}, j = 1, ..., d$.

If $\Lambda = \mathbb{R}_0$ then $M_j < \infty$ because $W(t,e_j)$ is continuous.

if $\Lambda = \mathbb{N}_0$ then $M_j < \infty$ because $||W(t,e_j)||_0 > 1$ only for finite number of values of t.

Let the norm of an initial condition $\bar\varphi$ be sufficiently small:

$||\bar\varphi||_0 < \varepsilon / \sum_{i=1}^d M_i$.

Extending $\bar\varphi$ in the basis we have

$\bar\varphi \equiv \sum_{j=1}^d \bar{x}_j e_j, ||\bar\varphi||_0 = \{\max|\bar{x}_j| : j = 1, ..., d\}$.

Hence, for all $t \in \Lambda$:

$$||W(t,\bar\varphi)||_0 = ||W(t, \textstyle\sum_{j=1}^d \bar{x}_j e_j)||_0 \le \textstyle\sum_{j=1}^d ||W(t,\bar{x}_j e_j)||_0 = \displaystyle\sum_{j=1}^d |\bar{x}_j| \cdot ||W(t,e_j)||_0$$

$$\le \{\max|\bar{x}_j| : j = 1, ..., d\} \sum_{j=1}^d ||W(t,e_j)||_0 \le (\varepsilon / \sum_{i=1}^d M_i) \sum_{j=1}^d M_j = \varepsilon.$$

The lemma is proven.

This result is not extended to *N-spaces*.

Example 3.1. Let $\Phi = Z$ be the set of finite sequences (or "infinite sequences with finite number of non-zero members") of real numbers with the basis $\{e_1, ..., e_d, ...\}$ and the norm $||\cdot||_0$. Consider the differential equation

$$w'(t) = diag\{(j - 2t) : j \in \mathbb{N}\} w(t), t \in \mathbb{R}_0.$$

Its general solution is $W(t, \sum_{j=1}^d x_j e_j) := \sum_{j=1}^d x_j \exp(jt - t^2) e_j$ with arbitrary d.

Then $||W(t, \sum_{j=1}^d x_j e_j)||_0 \le \max\{|x_j| : j = 1, ..., d\} \exp(d \cdot t - t^2) \to 0$ as $t \to \infty$, i.e. every solution is asymptotically equivalent to the zero solution.

On the other hand, for any $\varepsilon > 0$ and $t_1 > 0$ choose an integer number $d_1 > (-\log\varepsilon + t_1^2)/t_1$. Then we have

$||W(0, \varepsilon e_{d_1})||_0 = \varepsilon$;

$||W(t_1, \varepsilon e_{d_1})||_0 = \varepsilon \exp(d_1 t_1 - t_1^2) > \varepsilon \exp(-\log\varepsilon + t_1^2 - t_1^2) = 1$,

i.e. the zero solution is not stable.

Example 3.2. *1-spaces*: Considered the non-linear differential equation ($\Lambda = \mathbb{R}_0, \Phi = Z = \mathbb{R}$)

$$w'(t) = P(w(t)), t \in \mathbb{R}_0 \tag{2}$$

where $P(w)$ is a polynomial with coefficients in $\mathbb{R}$. Here $d^* = 0, \Phi'$ (Definition 2.4) is equivalent to the set of all stable real roots of $P(w)$; Φ^* (Definition 4) is equivalent to the set of all real roots of $P(w)$, (PARD($1 \searrow 0$)).

Example 3.3. (The Floquet-Lyapunov theory). *N-spaces*, $\Lambda = \mathbb{R}_0$. Some types of linear autonomous and periodical dynamical systems have finite or countable sets of characteristic values $\{\mu_1, \mu_2, ...\}$ which can be semi-ordered: $(Re(\mu_1) \geq Re(\mu_2) \geq ...\})$; (if such set is infinite then $\lim\{Re(\mu_k) : k \to \infty\} = -\infty$) such that functions $\exp(\mu_k t)$ (and for multiple values functions $\exp(\mu_k t)t^{p_k}, p_k \in \mathbb{N}$) are (components of) partial solutions.

If $(\forall\varphi \in \Phi)(\exists\{c_k\} \subset Z)(W(t, \varphi) = \Sigma_k c_k \exp(\mu_k t)t^{p_k})$ then: if $Re(\mu_1) > 0$ then the system is conditionally stable and if, additionally, $(\exists k)(Re(\mu_k) < 0)$ then PARD$_{(-Re\mu_k)}$ occurs.

If there are some $\mu_2, \mu_3, ...$ with $...0 > Re(\mu_2) > Re(\mu_3) > ...$ then multiple PARD$_{...-Re(\mu_2),-Re(\mu_3),...}$ occurs.

4. Special properties of delay differential equations

Since the 1950s some properties of solutions of initial value problems for differential equations with small delay were discovered and investigated. These properties had not analogs in other classes of dynamical systems.

One of the first publications was [6]. Reviews of results obtained in the 1970s are in [4] and [5]. A review of next results is in [7]. We propose to extend the discovered phenomena to wide classes of equations.

Considered *d-spaces*. Here $h > 0$ (the upper boundary for delay), $\Lambda = \mathbb{R}_h, Z = \mathbb{R}^d, \Phi = C([-h, 0] \to Z)$. Peculiarity of the properties of special solutions is that conditions on coefficients of equations are not on signs of corresponding characteristic values (as in Example 3.2) but on boundaries for coefficients.

We arrange the problems in enhancing order and extend them for *N-spaces*.

P1) Are the solutions $W(t, const)$ "slowly-varying"?

$((\exists\lambda_1 > 0)(\forall t_1, t_2 \in \Lambda)(||W(t_1, const)|| \geq ||W(t_2, const)|| \exp(-\lambda_1|t_1 - t_2|))$?

If they are then they are named *special ones*).

We also propose to call the set of initial values for special solutions (a subset of Φ) *a special set* and consider it as a self-standing object (see Definition 5.1 below).

P2) Does the equation have a unique solution for γ

$$W(t, \gamma) = W(t, \varphi(\cdot)) \tag{3}$$

for any $t \in \Lambda$ and $\varphi \in \Phi$?

If it has then denote the solution as $\Gamma(t, \varphi(\cdot))$; special solutions are said to be *representative.*

P3) Does $\Gamma^*(\varphi(\cdot)) := \lim\{\Gamma(t, \varphi(\cdot)) : t \to \infty\}$ exist?

If it is so then special solutions are said to be *approximating.*

P4) Does the difference $(W(t, \varphi(\cdot)) - W(t, \Gamma^*(\varphi(\cdot)))$ tend to zero?

If it does then special solutions are said to be *asymptotically approximating*; if $||W(t, const)||$ does not decrease then PARD($\infty \searrow d$) occurs.

P5) Is it true: $((\exists\lambda_2 > \lambda_1)(\forall\varphi \in \Phi)(\exists\gamma > 0)(||(W(t, \varphi(\cdot)) - W(t, \Gamma^*(\varphi(\cdot))|| \leq \gamma \exp(-\lambda_2 t))$?

If it is so then special solutions are said to be λ_2-*asymptotically approximating*; if $||W(t, const)||$ does not decrease then PARD$_{\lambda_2}(\infty \searrow d)$ occurs.

We will demonstrate obtained results by a scalar linear differential equation with constant delay $(\Lambda = \mathbb{R}_h, Z = \mathbb{R}, \Phi = C[-h, 0],)$

$$w'(t) = p(t)w(t-h), t \in \mathbb{R}_0;\ p(t) \in C(\mathbb{R}_0);\ (\forall t \in \mathbb{R}_0)(p(t) \in [p_-, p_+]) \tag{4}$$

with the initial condition

$$w(t) = \varphi(t) \in C[-h, 0],\ t \in [-h, 0]. \tag{5}$$

Theorem 4.1. [5]. Denote $p_0 := |[p_-, p_+]|$. If

$$\Delta := p_0 h < e^{-1} = 0.367... \tag{6}$$

(an absolute, dimensionless exact constant) then P1)-P5) occurs for solutions of the initial value problem (4)-(5); $0 < \lambda_1 < \lambda_2$ are solutions of the equation $\lambda = p_0 \exp(\lambda h)$.

We enlarged this result for an absolute domain in the two-dimensional space $\{p_- h, p_+ h\}$ (see Section 6 below).

We extended this theory (see the next section).

5. A class of difference equations with special solutions

In this section we describe a class of operator-difference equations with PARD$_\lambda$. Here $\Lambda = \mathbb{N}_0$. Let $\Phi = Z$ be a N-space. Considered the equation

$$w_0 \in \Phi;\quad w_{n+1} = F_n w_n, n \in \mathbb{N}_0 \tag{7}$$

where $F_n : \Phi \to \Phi$ are linear operators.

We will extend operators to sets (with same notations).

Definition 5.1. A connected closed set $\Phi_z \subset \Phi \setminus \{0\}$ is said to be *a special initial set* for the family of operators $\{F_n\}$ (for the equation (7)) if $(\exists q_- > 0)(\forall n \in \mathbb{N}_0)(F_n \Phi_z \subset q_- \Phi_z)$.

Lemma 5.1. For a special initial set Φ_z there exist such solutions $\{Z_n : n \in \mathbb{N}_0\}$ of (7) that $(\forall n \in \mathbb{N}_0)(Z_n \in q_-^n \Phi_z)$.

Proof. Choose the initial value $Z_0 \in \Phi_z$. By induction, due to linearity we have:
$Z_1 = F_0 Z_0 \subset F_0 \Phi_z \subset q_- \Phi_z$;
$Z_2 = F_1 Z_1 \subset F_1(q_- \Phi_z) = q_- F_1 \Phi_z \subset q_- q_- \Phi_z = q_-^2 \Phi_z, etc.$ The *lemma is proven by induction.*

We will call such solutions (together with the zero solution) special ones.

We proposed the following construction of *splitting the space* ([9, 10] [briefly). Introduce a linear projectional operator $P : \Phi \to \Phi$, denote $Q := I - P$ (also a linear projectional operator), Φ_x is the kernel of P, Φ_y is the image of P. (A subscript x or y will denote restriction of a linear operator defined on Φ to the subspace Φ_x or Φ_y and the corresponding norm.)

Denote variables $x_n := P z_n \in \Phi_x;\ y_n := Q z_n \in \Phi_y$ and operators
$a_n := PF_{xn} : \Phi_x \to \Phi_x; b_n := PF_{yn} : \Phi_y \to \Phi_x; c_n := QF_{xn} : \Phi_x \to \Phi_y; d_n := QF_{yn} : \Phi_y \to \Phi_y$.
Thus we obtain an initial value problem for a system

$$x_0 \in \Phi_x, y_0 \in \Phi_y;\quad x_{n+1} = a_n x_n + b_n y_n, y_{n+1} = c_n x_n + d_n y_n, n \in \mathbb{N}_0. \tag{8}$$

To formulate theorems denote the sets of operators (they are assumed to be bounded):

$(\forall n \in \mathbb{N}_0)(a_n \in A; b_n \in B; c_n \in C; d_n \in D)$.

We will use the denotation $||H||_-$ for a lower bound of a linear operator $H : ||Hx|| \geq ||H||_- ||x||$.

Theorem 5.1. If there exists such number $\eta > 0$ that

1) $q_- := ||A||_{x-} - \eta ||B||_x > 0$; 2)$||C||_y + \eta ||D||_y \leq \eta q_-$

then the set $\Phi_z = \{(x, y) \in \Phi_x \times \Phi_y : ||x||_x \geq 1; ||y||_y \leq \eta ||x||_x\}$ is a special initial set for the system (8).

Proof. Let $(x, y) \in \Phi_z$. Then we have for any n the estimation from below
$||F_n x||_x \geq ||Ax||_x - ||By||_x \geq ||A||_{x-}||x||_x - ||B||_x||y||_y \geq ||A||_{x-}||x||_x - \eta||B||_x||x||_x =$
$= (||A||_{x-} - \eta||B||_x)||x||_x = q_-||x||_x$
and the estimation
$||F_n y||_y \leq ||Cx||_y + ||Dy||_y \leq ||C||_y||x||_x + ||D||_y||y||_y \leq ||C||_y||x||_x + \eta||D||_y||x||_x =$
$= (||C||_y + \eta||D||_y)||x||_x \leq \eta q_-||x||_x$.
Hence, $||F_n y||_y/||F_n x||_x \leq \eta$. It means that $(F_n x, F_n y) \in q_-\Phi_z$.
The theorem is proven.

Corollary 5.1. The solution $\{(X_n, Y_n) : n \in \mathbb{N}_0\}$ of the system (8) with initial condition $X_n \neq 0, Y_0 = 0$ meets the conditions $(\forall n \in \mathbb{N})(||X_n||_x \geq q_-^n||X_0||_x; ||Y_n||_y \leq \eta||X_n||_x)$. As above, we will name solutions special.

Let $\Phi_1 = \mathbb{R}$. Then a_n are real numbers. We consider interval notations and restrictions on coefficients: let

$$(\forall n \in \mathbb{N}_0)(a_n \in [a_-, a_+]; ||b_n||_x \leq b_+ > 0; ||c_n||_y \leq c_+ > 0; ||d_n||_y \leq d_+ > 0). \tag{9}$$

It is seen that the product $\beta = b_+c_+$ is constant under linear variable substitution of y. Hence, we can pass to dimensionless variables, without loss of generality $b_+ = 1$ and we obtain the following corollary from Theorem 5.1.:

Theorem 5.2.. If $\Phi_x = \mathbb{R}$ and there exists such number $\xi > 0$ that
1) $q_- := a_- - \xi > 0$; 2) $\beta + \xi d_+ \leq \xi q_-$
then the set $\Phi_z = \{(x, y) \in \mathbb{R} \times \Phi_2 : x \geq 1; ||y||_y \leq \xi x\}$ is a special initial set; the space of corresponding special solutions of the system (8) is one-dimensional with the basic solution $\{(X_n, Y_n) : n \in \mathbb{N}_0\}$ meeting the following conditions:

$$X_0 := 1; Y_0 = 0; (\forall n \in \mathbb{N})(X_n \geq q_-^n; ||Y_n||_y \leq \xi X_n). \tag{10}$$

The following theorem provides sufficient conditions for the properties listed in Section 3.

Considered the system (8)-(9). Let $\{(x_n, y_n) : n \in \mathbb{N}_0\}$ be an arbitrary solution of (8) with initial values (x_0, y_0).

Theorem 5.3.
1) If $a_- - d_+ > 2\sqrt{\beta}$ then the conditions of Theorem 5.2. are satisfied and one may take
$\xi = \left(a_- - d_+ - \sqrt{(a_- - d_+)^2 - 4\beta}\right)/2$;
$q_- = \left(a_- + d_+ + \sqrt{(a_- - d_+)^2 - 4\beta}\right)/2$;
P1) the basic special solution (10) exists and P2) it is representative by the first component: $\Gamma_k(x_0, y_0) = x_k/X_k$.
2) If, additionally, $\omega := (a_+d_+ + \beta)q_-^{-2} < 1$ then
P3) the solution (10) is approximating by the first component: there exists a limit $\Gamma^*(x_0, y_0) := \lim\{\Gamma_k(x_0, y_0) : k \to \infty\}$;
3) If, additionally, $\omega_1 := \omega(a_+ + \xi) < 1$ then
P4) the solution (10) is asymptotically approximating by the first component: $\lim\{x_n - \Gamma^*(x_0, y_0)X_n : n \to \infty\} = 0$.

Proof. Denote $\zeta := \sqrt{(a_- - d_+)^2 - 4\beta}$ and *calculate*
$\beta + \xi d_+ - \xi q_- = \beta + (a_- - d_+ - \zeta)/2 \cdot (d_+ - (a_- + d_+ + \zeta)/2) = \beta - (a_- - d_+ - \zeta)(a_- - d_+ + \zeta)/4 = \beta - ((a_- - d_+)^2 - \zeta^2)/4 = 0$.

Hence, the numbers ξ and q_- fulfil the equality $\beta + \xi d_+ = \xi q_-$. *The conditions of Theorem 5.2. are satisfied, the basic special solution (10) exists. For any k we have $\Gamma_k(x_0, y_0)(X_n, Y_n)|_{n=k} = x_k/X_k \cdot (X_k, Y_k) = (x_k, x_k/X_k \cdot Y_k)$. Statement 1) is proven.*

P*roof of statement 2).* Denote $\Omega_n := a_n d_n - c_n b_n : \Phi_y \to \Phi_y$. Transform (Casorati determinant into discrete analog of Wronskian determinant):

$$|\Gamma_{n+1}(x_0, y_0) - \Gamma_n(x_0, y_0)| = |x_{n+1}/X_{n+1} - x_n/X_n| = |x_{n+1}X_n - x_n X_{n+1}|/(X_{n+1}X_n) =$$

$$= |b_n(X_n y_n - x_n Y_n)|/(X_{n+1}X_n) = |b_n \Pi_{k=0}^{n-1}\Omega_k(X_0 y_0 - x_0 Y_0)|/(X_{n+1}X_n). \qquad (11)$$

We have for all $n \in \mathbb{N}_0$: $||\Omega_n||_y \le ||a_n d_n||_y + ||c_n b_n||_y \le a_+ d_+ + \beta = \omega q_-^2$. *The equality (11) yields the following estimation* (*const stand for constants that do not depend on* n):

$|\Gamma_{n+1}(x_0, y_0) - \Gamma_n(x_0, y_0)| \le ||b_n||_x \Pi_{k=0}^{n-1}||\Omega_k||\cdot||y_0||_2/(X_{n+1}X_n) \le b_+||y_0||_y(\omega q_-^2)^n/(X_{n+1}X_n) \le const \cdot (\omega q_-^2)^n/(q_-^{n+1}q_-^n) = const_1 \cdot \omega^n$.

Hence, the sequence $\{\Gamma_n(x_0, y_0) : n \in \mathbb{N}\}$ *converges and has a limit* $\Gamma^*(x_0, y_0)$. By the known estimation, $|\Gamma^*(x_0, y_0) - \Gamma_n(x_0, y_0)| \le const_1 \cdot \omega^n/(1-\omega)$.

Statement 2) is proven.

Proof of statement 3). Estimate $X_{n+1} = a_n X_n + b_n Y_n \le a_+ X_n + b||Y_n||_y \le (a_+ + \xi)X_n$. Hence, $(\forall n \in \mathbb{N})(X_n \le (a_+ + \xi)^n)$.

$|x_n - \Gamma^*(x_0, y_0)X_n| = |x_n/X_n - \Gamma^*(x_0, y_0)|X_n \le$

$|\Gamma_n(x_0, y_0) - \Gamma^*(x_0, y_0)|(a_+ + \xi)^n \le const_2 \cdot \omega^n (a_+ + \xi)^n = const_2 \cdot \omega_1^n$

tends to zero as $n \to \infty$.

Statement 3) is proven.

The theorem is proven.

The condition 1) in Theorem 5.2. is close to necessary one. Consider the system in $\mathbb{R}^2$ with constant coefficients:

$x_{n+1} = a_- x_n - y_n, y_{n+1} = \beta x_n + d_+ y_n, n \in \mathbb{N}_0$,

the characteristic equation is $\lambda^2 - (a_- + d_+)\lambda + a_- d + \beta = 0$.

If $a_- - d_+ < 2\sqrt{\beta}$ then $(a_- + d_+)^2 - 4(a_- d_+ + \beta) = (a_- - d_+)^2 - 4\beta < 0$ and the characteristic numbers are complex. Hence, all non-zero solutions of the system are oscillating and special solutions absent.

The conditions of the Theorem 5.2. are not easy to verify precisely. However, by means of a computer program, we have obtained the following result numerically. The program with directed rounding was written in *PASCAL*.

Theorem 5.4. If $0.9 \le a_n \le 1.9, bc \le 0.06, d \le 0.3$ then P1)-P2); if $0.9 \le a_n \le 1.7$ then P3); if $0.9 \le a_n \le 1.2$ then P4) (by the first argument).

6. Improving of results on delay differential equations

In this section we apply Theorem 5.2. to the delay differential equation (4). We have ascertained that the condition of "small delay" for linear delay differential equations of type (4) and more general ones corresponds to the conditions "the interval $[a_-, a_+]$ is close to 1; the positive numbers $b_+ c_+$ and d_+ are small" for coefficients in (9).

We see that $w'(t)$ exists for $t \ge 0$, hence without loss of generality we assume that $\varphi \in C^1[-h, 0]$ in (5).

The shift operator for the equation (4)

$$Sw(\cdot)(t) := w(0) + \int_{-h}^{t} p(s)w(s)ds, t \in [-h, 0] \qquad (12)$$

with corresponding shifts of the argument yields the solution $w(t)$ consequently on intervals $[0, h], [h, 2h], ...$ The forthcoming transformations and notations are also valid for shifts of the argument.

Split the space $\Phi := C^1[-h, 0] = \mathbb{R} \times \Phi_2 \ni w = w(0) + (w(t) - w(0)) \equiv x + y(t);\ y(0) = 0$. Let Φ_2 be the set of functions $\{y(t) \in C^1[-h, 0] : y(0) = 0\}$ with the norm $||y||_2 := sup\{|y'(t)| : t \in [-h, 0]\}$, then $|y(t)| \leq ||y||_2|t|$.

The shift operator (12) can also be splitted:

$$S(x + y(\cdot))(t) := x(0) + \int_{-h}^{0} p(s)(x + y(s))ds + \int_{0}^{t} p(s)(x + y(s))ds.$$

Denoting operators on each segment of length h we obtain:

$$\tilde{a}x := (1 + \int_{-h}^{0} p(s)ds)x : \mathbb{R} \to \mathbb{R}; \quad \tilde{b}y(\cdot) := \int_{-h}^{0} p(s)y(s)ds : \Phi_2 \to \mathbb{R};$$

$$(\tilde{c}x)(t) := \int_{0}^{t} p(s)ds\ x : \mathbb{R} \to \Phi_2; \quad (\tilde{d}y(\cdot))(t) := \int_{0}^{t} p(s)y(s)ds : \Phi_2 \to \Phi_2.$$

Thus, the equation (4) can be written in the form (8).

Estimating these integrals we obtain for constants in (9)

$$a_- = 1 - \Delta; a_+ = 1 + \Delta; b_+ = \Delta h/2; c_+ = p_0; d_+ = \Delta/2, \beta = \Delta^2/2.$$

Substituting these estimations in Theorem 5 by means of a computer program with directed rounding in *PASCAL* we have proven

Theorem 6.1. If $\Delta < 0.343$ then the condition 1) of Theorem 5.2. fulfils; If $\Delta < 0.327$ then the condition 2) fulfils; If $\Delta < 0.304$ then the condition 3) fulfils.

These results illustrate that the method of Section 5 can be applied to delay differential equations. The results of Theorem 6.1. are similar to ones of [6] but are weaker than the exact one (6).

This method can also yield better results.

Denote $\Delta_- := p_- h, \Delta_+ := p_+ h$. Then we obtain for constants in (9)

$$a_- = 1 + \Delta_-; a_+ = 1 + \Delta_+; b_+ = \Delta_+ h/2; c_+ = p_+; d_+ = \Delta_+/2, \beta = \Delta_+^2/2.$$

Substituting in Theorem 5 by means of a computer program with directed rounding in *PASCAL* we have proven

Theorem 6.2. If either $-0.05 \leq p(t)h \leq 0.42$ or $-0.040 \leq p(t)h \leq 0.43$ or $-0.01 \leq p(t)h \leq 0.44$ then special solutions of the equation (4) exist and are asymptotically approximating.

These results present any absolute domain and enlarge the domain (6) for coefficients of delay differential equations providing special properties.

7. Conclusion

The paper demonstrates that many results on asymptotic behavior of solutions to dynamic systems can be stated uniformly by means of the new notions "asymptotic equivalence" and "asymptotic reduction of solution space dimension". The new method of splitting spaces provides extension of the phenomenon of special solutions onto large classes of operator difference equations and obtaining new results for delay differential equations.

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